FULL DETERMINATION OF THE CHARACTERISTICS OF ELASTIC SCATTERS FROM SOME FFP MEASUREMENTS

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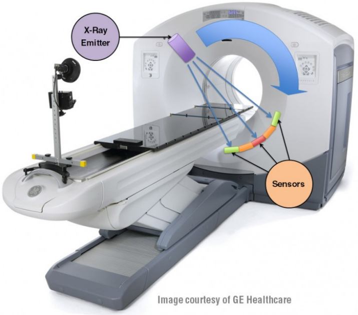
Main Goal

Recover the shape and material parameters of

an elastic object in a fluid

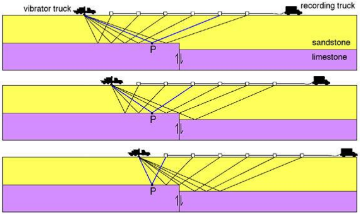
Motivation

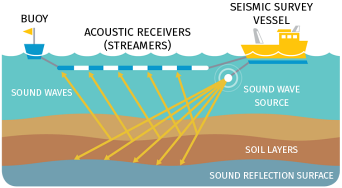
Obstetric ultrasonography



X-ray computed tomography

Motivation

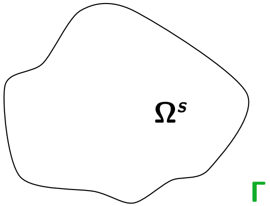




Terrestrial seismic survey

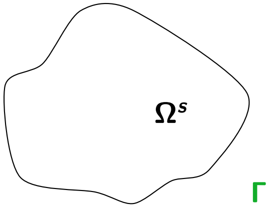
Marine seismic survey

Mathematical model



Mathematical model

$$\Omega^f = \mathbb{R}^2 \setminus \overline{\Omega}^s$$



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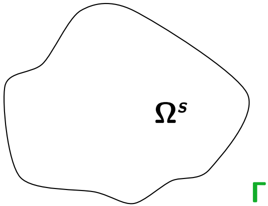
$$\Omega^f = \mathbb{R}^2 \setminus \overline{\Omega}^s$$

- **Source**



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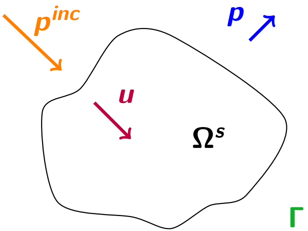
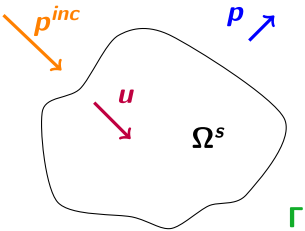


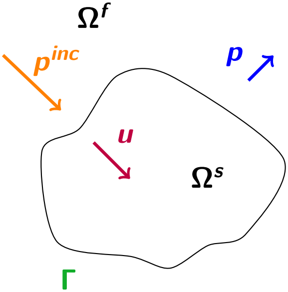
Mathematical model

$$\Omega^f = \mathbb{R}^2 \setminus \overline{\Omega}^s$$

- Source
- Receiver



Mathematical model

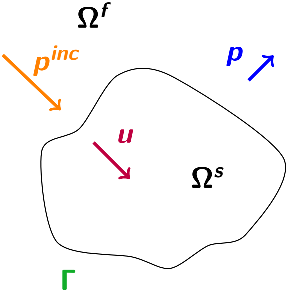


Mathematical model

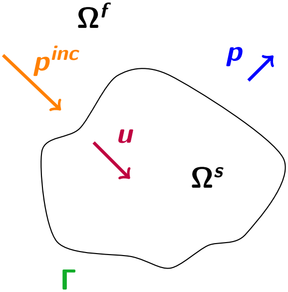
$$\nabla \cdot \sigma(u) \quad + \quad \omega^2 \rho_s u = 0 \qquad \text{in } \Omega^s$$

$$\Delta p \quad + \quad k^2 p = 0 \qquad \text{in } \Omega^f$$

$$k = \left(\frac{\omega}{c_f}\right)$$



Mathematical model



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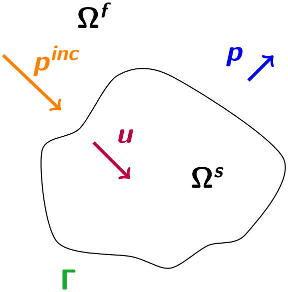
$$\omega^2 \rho_f u \cdot \nu = \frac{\partial p}{\partial \nu} + \frac{\partial p^{inc}}{\partial \nu} \qquad \text{on } \Gamma$$

$$\sigma(u)\nu = -p\nu - p^{inc} \qquad \text{on } \Gamma$$

Mathematical model



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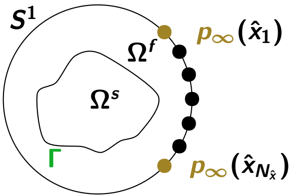
$$\sigma(u)\nu = -p\nu - p^{inc} \quad \text{on } \Gamma$$

$$\lim_{r \to +\infty} \sqrt{r}\left(\frac{\partial p}{\partial r} - ikp\right) = 0 \quad (r = \| x \|_2)$$

Mathematical Model

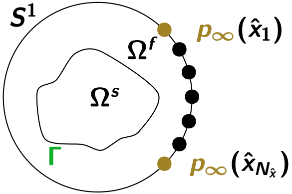
The Far-Field Pattern (FFP)



$$p_\infty(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_\Gamma \left(e^{ik\hat{x}\cdot y} \frac{\partial p}{\partial \nu}(y) - \frac{\partial e^{ik\hat{x}\cdot y}}{\partial \nu} p(y) \right) dy$$

Mathematical Model

The Far-Field Pattern (FFP)



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FFP Intensity

$$| \, p_\infty(\hat{x}) \, | = \sqrt{\bar{p}_\infty(\hat{x}) \cdot p_\infty(\hat{x})}$$

The Inverse Problem

- **Direct Mapping**

$$(\lambda, \mu, \Gamma) \longmapsto A(\lambda, \mu, \Gamma)(\hat{x}) = \mid p_\infty(\hat{x}) \mid$$

The Inverse Problem

- **Direct Mapping**

$$(\lambda, \mu, \Gamma) \longmapsto A(\lambda, \mu, \Gamma)(\hat{x}) = \mid p_\infty(\hat{x}) \mid$$

- **Inverse Problem**

$$\left\{ \begin{array}{l} \text{Given } \mid p_\infty(x_j) \mid, \text{ find } (\lambda, \mu, \Gamma), \text{ such that} \\ A(\lambda, \mu, \Gamma)(x_j) = \mid p_\infty(x_j) \mid; \quad j = 1, \ldots, M \end{array} \right.$$

The Inverse Problem

- **Direct Mapping**

$$(\lambda, \mu, \Gamma) \longmapsto A(\lambda, \mu, \Gamma)(\hat{x}) = | \, p_\infty(\hat{x}) \, |$$

- **Inverse Problem**

$$\begin{cases} \text{Given } \; | \, p_\infty(x_j) \, |, \; \text{find } (\lambda, \mu, \Gamma), \; \text{such that} \\ A(\lambda, \mu, \Gamma)(x_j) = | \, p_\infty(x_j) \, |; \quad j = 1, ..., M \end{cases}$$

$\Longrightarrow$ **Nonlinear & Ill-Posed problem**

The Nonlinearity Issue

$$\min_{\lambda,\mu,\Gamma} \frac{1}{2} || \sum_{j=1}^{N_x} A\left(\lambda,\mu,\Gamma\right)\left(\hat{x}_j\right) - \left| p_\infty(\hat{x}_j) \right| ||_2^2$$

The Nonlinearity Issue

The Newton Equation

$$J^{(n)}(\hat{x}_j) \cdot \begin{pmatrix} \delta\lambda^{(n)} \\ \delta\mu^{(n)} \\ \delta\Gamma^{(n)} \end{pmatrix} = \mid p_\infty(\hat{x}_j) \mid - A\left(\lambda^{(n)}, \mu^{(n)}, \Gamma^{(n)}\right)(\hat{x}_j)$$

$$j = 1, ..., N_x$$

The Nonlinearity Issue

The Newton Equation

$$J^{(n)}(\hat{x}_j) \cdot \begin{pmatrix} \delta\lambda^{(n)} \\ \delta\mu^{(n)} \\ \delta\Gamma^{(n)} \end{pmatrix} = \mid p_\infty(\hat{x}_j) \mid - A\left(\lambda^{(n)}, \mu^{(n)}, \Gamma^{(n)}\right)(\hat{x}_j)$$

$$j = 1, \ldots, N_x$$

Update the Parameters

$$(\lambda^{(n+1)}, \mu^{(n+1)}, \Gamma^{(n+1)}) = (\lambda^{(n)}, \mu^{(n)}, \Gamma^{(n)}) + (\delta\lambda^{(n)}, \delta\mu^{(n)}, \delta\Gamma^{(n)})$$

The Nonlinearity Issue

Least-Squares Formulation

$$B^{(n)} \cdot \delta^{(n)} \quad = f^{(n)}$$

$$B^{(n)} = J^{*(n)}(\hat{x}) \cdot J^{(n)}(\hat{x})$$

$$f^{(n)} = J^{*(n)}(\hat{x}) \left(\mid p_{\infty}(\hat{x}_j) \mid - A\left(\lambda^{(n)}, \mu^{(n)}, \Gamma^{(n)}\right)(\hat{x}) \right)$$

The Stability Issue

The Regularized Newton Equation

$$B^{(n)}\delta^{(n)} \quad + \quad \alpha^{*(n)}\delta^{(n)} = f^{(n)}$$

$$\alpha^{*(n)} = \begin{pmatrix} \alpha_m^{(n)} & 0 & 0 \\ 0 & \alpha_m^{(n)} & 0 \\ 0 & 0 & \alpha_s^{(n)} \end{pmatrix}$$

How to compute the derivatives?

Derivative with respect to Γ

Derivative with respect to Γ

$$
\begin{aligned}
\nabla \cdot \sigma(u') + \omega^2 \rho_s u' &= 0 && \text{in } \Omega^s \\
\Delta p' + k^2 p' &= 0 && \text{in } \Omega^f \\
\sigma(u') \cdot \nu + p' \cdot \nu &= F_j(u, p, h_j) && \text{on } \Gamma \\
\omega^2 \rho_s u' \cdot \nu - \partial_\nu p' &= G_j(u, p, h_j) && \text{on } \Gamma
\end{aligned}
$$

$$
\lim_{r \to +\infty} \sqrt{r} \left(\frac{\partial p'}{\partial r} - ikp' \right) = 0
$$

Derivative with respect to Γ

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\nabla \cdot \sigma(u') + \omega^2 \rho_s u' &= 0 && \text{in } \Omega^s \\
\Delta p' + k^2 p' &= 0 && \text{in } \Omega^f \\
\sigma(u') \cdot \nu + p' \cdot \nu &= F_j(u, p, h_j) && \text{on } \Gamma \\
\omega^2 \rho_s u' \cdot \nu - \partial_\nu p' &= G_j(u, p, h_j) && \text{on } \Gamma \\
\lim_{r \to +\infty} \sqrt{r} \left(\frac{\partial p'}{\partial r} - ikp' \right) &= 0
\end{aligned}
$$

$$
\begin{aligned}
F_j(u, p, h_j) &= -h_j^t \nabla \sigma(u) \nu - \nabla p^T \cdot h_j \nu + \sigma(u)[h_j']^t \nu + p^T [h_j']^t \nu \\
G_j(u, p, h_j) &= -(\omega^2 \rho_f \nabla u - \nabla(\nabla p^T)) h_j \cdot \nu + (\omega^2 \rho_f u v - \nabla p^T) \cdot [h_j']^t \nu
\end{aligned}
$$

Derivative with respect to the material parameters

Derivative with respect to the material parameters

$$
\begin{aligned}
\nabla \cdot \sigma(\mathbf{u}') + \omega^2 \rho_s \mathbf{u}' &= - \nabla \cdot \left(\mathbf{C}' \varepsilon(\mathbf{u}) \right) && \text{in } \Omega^s \\
\Delta \mathbf{p}' + k^2 \mathbf{p}' &= 0 && \text{in } \Omega^f \\
\sigma(\mathbf{u}') \cdot \nu + \mathbf{p}' \cdot \nu &= \mathbf{C}' \varepsilon(\mathbf{u}) \cdot \nu && \text{on } \Gamma \\
\omega^2 \rho_s \mathbf{u}' \cdot \nu + \partial_\nu \mathbf{p}' &= 0 && \text{on } \Gamma
\end{aligned}
$$

$$
\lim_{r \to +\infty} \sqrt{r} \left(\frac{\partial \mathbf{p}'}{\partial r} - ik\mathbf{p}' \right) = 0
$$

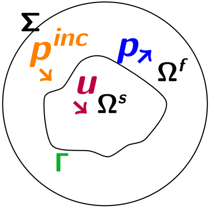
How to solve the direct problem?

A DG-type Solution Methodology

Main Features

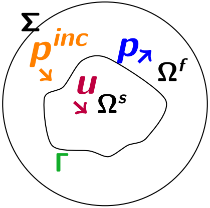
A DG-type Solution Methodology

Main Features



A DG-type Solution Methodology

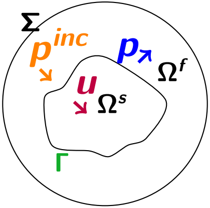
Main Features



- **Discontinuous Galerkin method with Interior Penalty (IPDG)**

A DG-type Solution Methodology

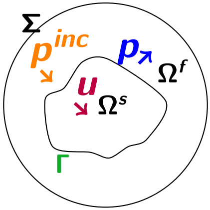
Main Features



- **Discontinuous Galerkin method with Interior Penalty (IPDG)**
- **Higher-order elements**

A DG-type Solution Methodology

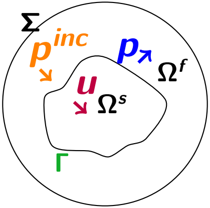
Main Features



- Discontinuous Galerkin method with Interior Penalty (IPDG)
- Higher-order elements
- Curved edges on the boundaries Γ and Σ

Solution Methodology for the Direct Scattering problem

Algebraic Formulation



$$\begin{pmatrix} A^f & B \\ B^* & A^s \end{pmatrix} \begin{pmatrix} P \\ U \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

Solution Methodology for the Inverse Problem

Main Features

Solution Methodology for the Inverse Problem

Main Features

- **Recognize λ, μ, Γ are of different nature**

Solution Methodology for the Inverse Problem

Main Features

- **Recognize λ, μ, Γ are of different nature**
  - **Rescaling procedure**

Solution Methodology for the Inverse Problem

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  - **Prediction/Correction type approache**

Solution Methodology for the Inverse Problem

Main Features

- Recognize λ, μ, Γ are of different nature
  - Rescaling procedure
  - Prediction/Correction type approache
- Recognize the influence of the frequency on the scattering observability

Solution Methodology for the Inverse Problem

Main Features

- Recognize λ, μ, Γ are of different nature
  - Rescaling procedure
  - Prediction/Correction type approache
- Recognize the influence of the frequency on the scattering observability
  - Switch to a higher frequency

Solution methodology for the inverse problem

Multistage Solution Procedure

- **Stage 0, Parameters Initialization**

Solution methodology for the inverse problem

Multistage Solution Procedure

- **Stage 0,** Parameters Initialization

  $\lambda^{(0)}$, $\mu^{(0)}$, $\Gamma^{(0)}$

Solution methodology for the inverse problem

Multistage Solution Procedure

- **Stage 0,** **Parameters Initialization**

  $\lambda^{(0)}$, $\mu^{(0)}$, $\Gamma^{(0)}$

- **Stage 1,** **Fix the frequency**

Solution methodology for the inverse problem

Multistage Solution Procedure

- **Stage 0,** Parameters **Initialization**

  $\lambda^{(0)}, \mu^{(0)}, \Gamma^{(0)}$

- **Stage 1,** **Fix** the frequency
  - **Apply regularized Newton algorithm until convergence or stagnation**

Solution methodology for the inverse problem

Multistage Solution Procedure

- **Stage 0,** Parameters Initialization

  $\lambda^{(0)}$, $\mu^{(0)}$, $\Gamma^{(0)}$

- **Stage 1,** Fix the frequency
  - Apply regularized Newton algorithm until convergence or stagnation
  - If stagnation, go to Stage 2.

Solution methodology for the inverse problem

Multistage Solution Procedure

- **Stage 0,** Parameters Initialization

  $\lambda^{(0)}$, $\mu^{(0)}$, $\Gamma^{(0)}$

- **Stage 1,** Fix the frequency
  - Apply regularized Newton algorithm until convergence or stagnation
  - If stagnation, go to Stage 2.
- **Stage 2,** Switch to a higher frequency and repeat Stage 1.

Solution methodology for the inverse problem

Computational Complexity

- **Step 0,** **Initialization** $| \, p_\infty \, |$

Solution methodology for the inverse problem

Computational Complexity

- **Step 0, Initialization** $\mid p_\infty \mid$
  - **Solve one forward problem**

Solution methodology for the inverse problem

Computational Complexity

- **Step 0,** **Initialization $|\ p_\infty\ |$**
  - **Solve one forward problem**
- **Step 1,** **Construction of $J^{(n)}$**

Solution methodology for the inverse problem

Computational Complexity

- **Step 0,** Initialization $| p_\infty |$
  - Solve one forward problem
- **Step 1,** Construction of $J^{(n)}$
  - Solve same forward problem with P right-hand sides

Solution methodology for the inverse problem

Computational Complexity

- **Step 0,** **Initialization** $|\, p_\infty \,|$
  - **Solve one forward problem**
- **Step 1,** **Construction of** $J^{(n)}$
  - **Solve same forward problem with P right-hand sides**
- **Step 2,** **Solve regularized Newton equation**

Solution methodology for the inverse problem

Computational Complexity

- **Step 0,** Initialization $| p_\infty |$
  - Solve one forward problem
- **Step 1,** Construction of $J^{(n)}$
  - Solve same forward problem with P right-hand sides
- **Step 2,** Solve regularized Newton equation
  - Solve one PxP linear system

Illustrative Numerical Results

- **Circular-Shaped Domain**

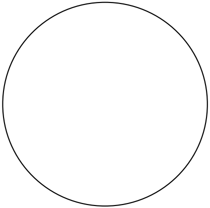
Illustrative Numerical Results

- **Circular-Shaped Domain**
- **Polygonal-Shaped Domain**

Circular-Shaped Domain

Steel Disk Domain

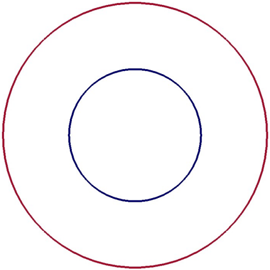
Data



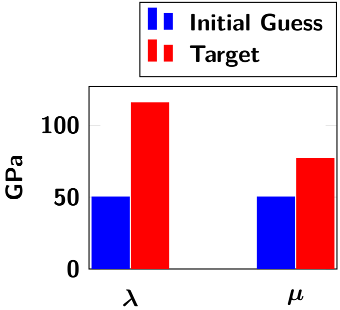
- $\rho_s = 7900 kg \cdot m^{-1}$
- $r = 1cm$
- $\lambda = 115.4$ GPa
- $\mu = 76.9$ GPa
- $f = 63.66$ kHz
- Noise level=0%
- Observations: 360 measurement points (full-aperture)

Steel Disk Domain

Initial data



(a) **Shape parameters**

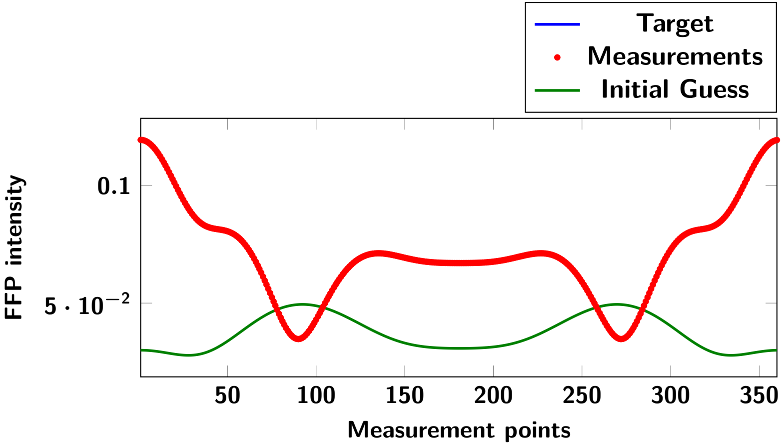


(b) **Material parameters**

Initial Relative Error on shape parameter 50%, on material parameters 51%.

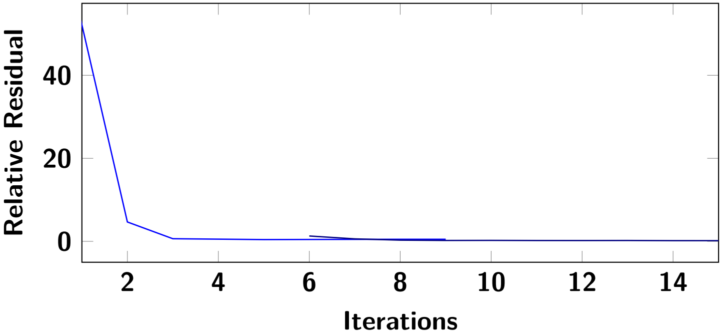
Steel Disk Domain

Initial Relative Residual: 114.28%



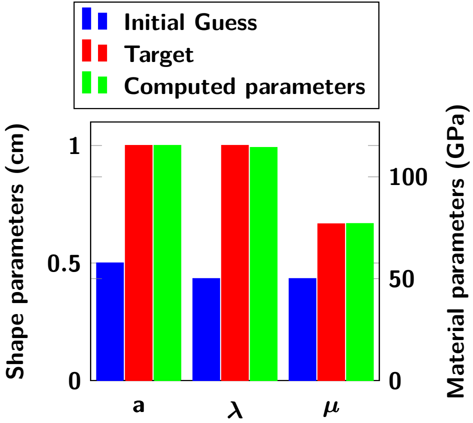
Steel Disk Domain

Convergence history



Steel Disk Domain

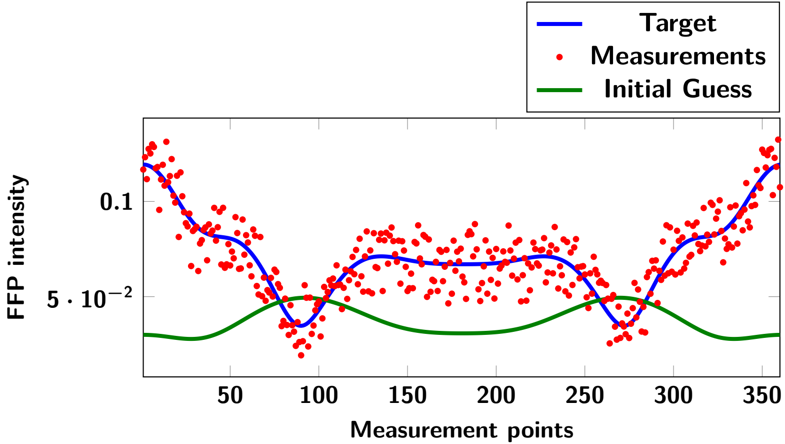
Accuracy



Relative error on shape parameter 0.01%, on material parameter 0.73%.

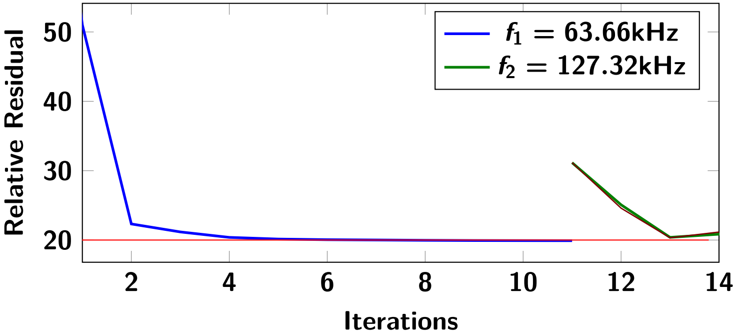
Steel Disk Domain, 20% noise-level

Initial Relative Residual: 112.59%



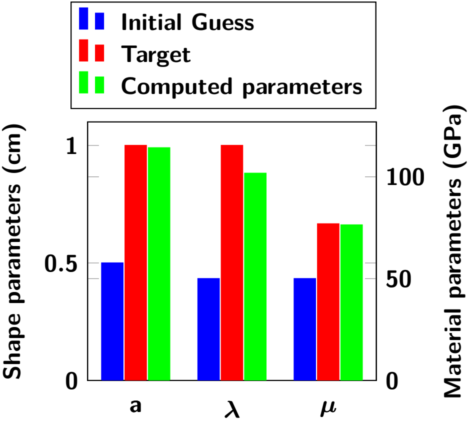
Steel Disk Domain, 20% noise-level

Convergence history



Steel Disk Domain, 20% noise-level

Accuracy

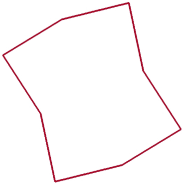


Relative Error on shape parameter 0.4%, on material parameter 10.33%.

Steel Polygonal-Shaped Domain

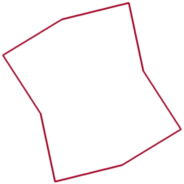
Steel Polygonal-Shaped Domain

Data



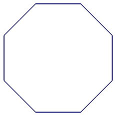
Steel Polygonal-Shaped Domain

Data

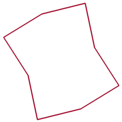


- $\rho_s = 7900 kg \cdot m^{-1}$
- $r = 1 cm$
- $\lambda = 115.4$ GPa
- $\mu = 76.9$ GPa
- $f_1 = 63.66$ kHz, $f_2 = 143.24$ kHz
- Noise level=5%
- Observations: 360 measurement points (full-aperture)

Steel Polygonal-Shaped Domain



(a) Initial Guess

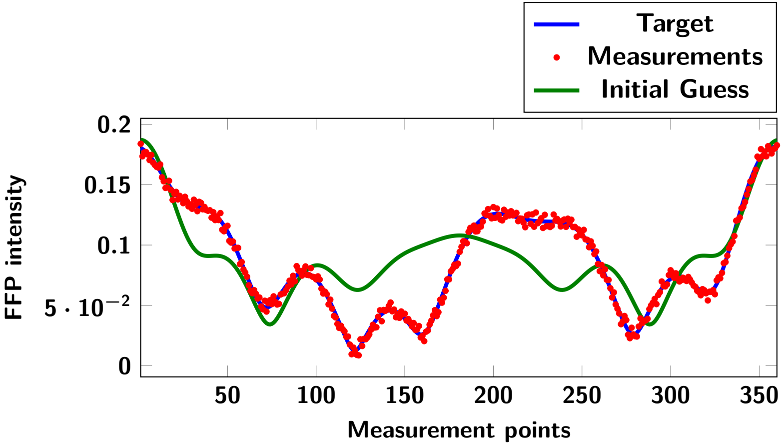


(b) Target

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
Target	**1.06**	**1.84**	**1.50**	**1.84**	**1.06**	**1.84**	**1.50**	**1.84**
Initial guess	**1.75**	**1.75**	**1.75**	**1.75**	**1.75**	**1.75**	**1.75**	**1.75**
Relative Error	**64.98**	**4.89**	**16.67**	**4.89**	**64.98**	**4.89**	**16.67**	**4.89**

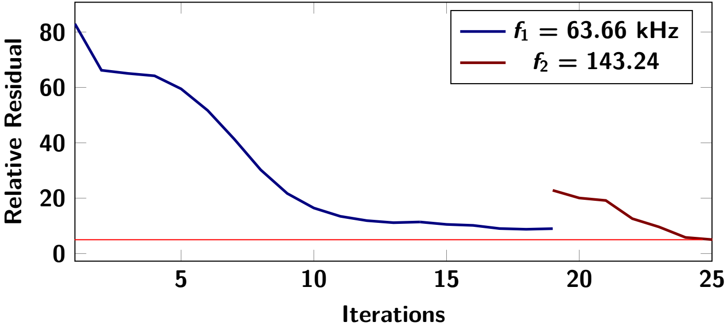
Steel Polygonal-Shaped Domain

Initial Relative Residual: 82.54%



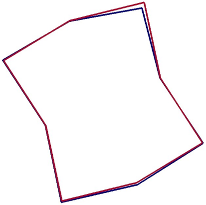
Steel Polygonal-Shaped Domain

Convergence history



Polygonal-shaped domain, 5% noise-level

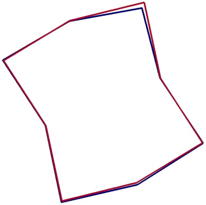
Accuracy

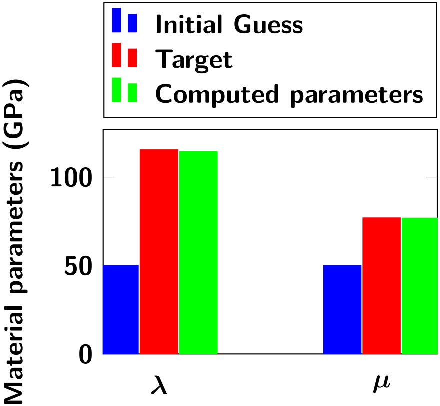


Relative Errors on shape parameters 0.66%, on material parameters 0.75%.

Polygonal-shaped domain, 5% noise-level

Accuracy

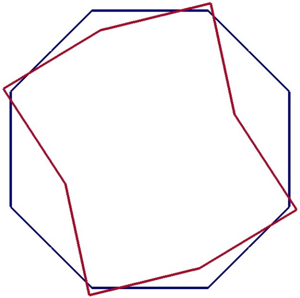




Relative Errors on shape parameters 0.66%, on material parameters 0.75%.

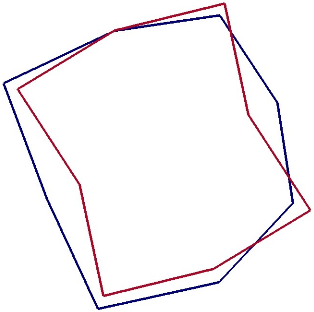
Steel Polygonal-Shaped Domain

$\#Iter = 0$



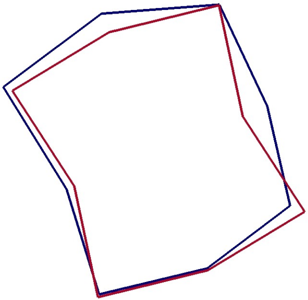
Steel Polygonal-Shaped Domain

$\#Iter = 3$



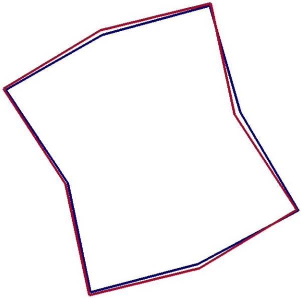
Steel Polygonal-Shaped Domain

$\#Iter = 8$



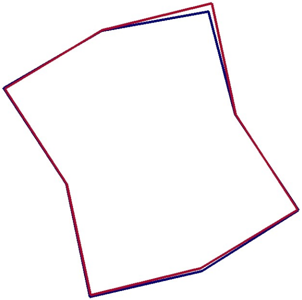
Steel Polygonal-Shaped Domain

$\#Iter = 17$



Steel Polygonal-Shaped Domain

$\#Iter = 23$



Conclusions and perspectives

- Promising results

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- Pursue numerical experiments

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- Promising results
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- Extension to 3D models

Thank you for listening